11-8 Proof and Mathematical Induction

Main Ideas

- Prove statements by using mathematical induction.
- Disprove statements by finding a counterexample.

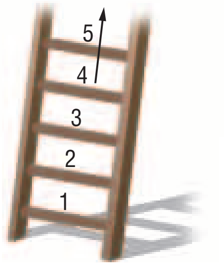
New Vocabulary

mathematical induction

inductive hypothesis

GET READY for the Lesson

Imagine the positive integers as a ladder that goes upward forever. You know that you cannot leap to the top of the ladder, but you can stand on the first step, and no matter which step you are on, you can always climb one step higher. Is there any step you cannot reach?



Mathematical Induction **Mathematical induction** is used to prove statements about positive integers. This proof uses three steps.

> **KEY CONCEPT** — *Mathematical Induction*
>
> **Step 1** Show that the statement is true for some positive integer n.
>
> **Step 2** Assume that the statement is true for some positive integer k, where $k \geq n$. This assumption is called the **inductive hypothesis.**
>
> **Step 3** Show that the statement is true for the next positive integer $k + 1$. If so, we can assume that the statement is true for any positive integer.

> **Study Tip**
>
> **Step 1**
> In many cases, it will be helpful to let $n = 1$.

EXAMPLE 1 Summation Formula

Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Step 1 When $n = 1$, the left side of the given equation is 1^2 or 1.

The right side is $\frac{1(1+1)[2(1)+1]}{6}$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume $1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$ for a positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{Add } (k+1)^2 \text{ to each side.}$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \qquad \text{Add.}$$
$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \qquad \text{Factor.}$$
$$= \frac{(k+1)[2k^2 + 7k + 6]}{6} \qquad \text{Simplify.}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6} \qquad \text{Factor.}$$
$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

The last expression is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus, the equation is true for $n = k + 1$. This proves the conjecture.

CHECK Your Progress

1. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

EXAMPLE Divisibility

2 **Prove that $7^n - 1$ is divisible by 6 for all positive integers n.**

Step 1 When $n = 1$, $7^n - 1 = 7^1 - 1$ or 6. Since 6 is divisible by 6, the statement is true for $n = 1$.

Step 2 Assume that $7^k - 1$ is divisible by 6 for some positive integer k. This means that there is a whole number r such that $7^k - 1 = 6r$.

Step 3 Show that the statement is true for $n = k + 1$.

$7^k - 1$	$= 6r$	Inductive hypothesis
7^k	$= 6r + 1$	Add 1 to each side.
$7(7^k)$	$= 7(6r + 1)$	Multiply each side by 7.
7^{k+1}	$= 42r + 7$	Simplify.
$7^{k+1} - 1$	$= 42r + 6$	Subtract 1 from each side.
$7^{k+1} - 1$	$= 6(7r + 1)$	Factor.

Since r is a whole number, $7r + 1$ is a whole number. Therefore, $7^{k+1} - 1$ is divisible by 6. Thus, the statement is true for $n = k + 1$.

This proves that $7^n - 1$ is divisible by 6 for all positive integers n.

CHECK Your Progress

2. Prove that $10^n - 1$ is divisible by 9 for all positive integers n.

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Review Vocabulary

Counterexample
a specific case that shows that a statement is false (Lesson 1-2)

Counterexamples Of course, not every equation that you can write is true. You can show that an equation is not always true by finding a *counterexample*.

EXAMPLE Counterexample

3 **Find a counterexample for $1^4 + 2^4 + 3^4 + \cdots + n^4 = 1 + (4n - 4)^2$.**

n	Left Side of Formula	Right Side of Formula	
1	1^4 or 1	$1 + [4(1) - 4]^2 = 1 + 0^2$ or 1	true
2	$1^4 + 2^4 = 1 + 16$ or 17	$1 + [4(2) - 4]^2 = 1 + 4^2$ or 17	true
3	$1^4 + 2^4 + 3^4 = 1 + 16 + 81$ or 98	$1 + [4(3) - 4]^2 = 1 + 64$ or 65	false

The value $n = 3$ is a counterexample for the equation.

CHECK Your Progress

3. Find a counterexample for the statement that $2n^2 + 11$ is prime for all positive integers n.

CHECK Your Understanding

Example 1 (pp. 670–671)

Prove that each statement is true for all positive integers.

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

2. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

3. PARTIES Suppose that each time a new guest arrives at a party, he or she shakes hands with each person already at the party. Prove that after n guests have arrived, a total of $\frac{n(n-1)}{2}$ handshakes have taken place.

Example 2 (p. 671)

Prove that each statement is true for all positive integers.

4. $4^n - 1$ is divisible by 3.

5. $5^n + 3$ is divisible by 4.

Example 3 (p. 672)

Find a counterexample for each statement.

6. $1 + 2 + 3 + \cdots + n = n^2$

7. $2^n + 3^n$ is divisible by 4.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8–11	1
12, 13	2
14, 15	1, 2
16–21	3

Prove that each statement is true for all positive integers.

8. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

9. $2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n+1)}{2}$

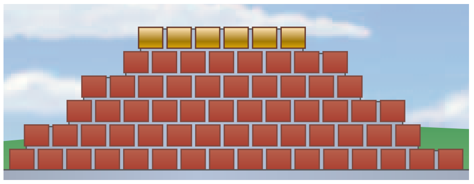
10. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

11. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

12. $8^n - 1$ is divisible by 7.

13. $9^n - 1$ is divisible by 8.

14. ARCHITECTURE A memorial being constructed in a city park will be a brick wall, with a top row of six gold-plated bricks engraved with the names of six local war veterans. Each row has two more bricks than the row above it. Prove that the number of bricks in the top n rows is $n^2 + 5n$.



15. NATURE The terms of the Fibonacci sequence are found in many places in nature. The number of spirals of seeds in sunflowers are Fibonacci numbers, as are the number of spirals of scales on a pinecone. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, ... Each element after the first two is found by adding the previous two terms. If f_n stands for the nth Fibonacci number, prove that $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$.

Find a counterexample for each statement.

16. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(3n-1)}{2}$

17. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = 12n^3 - 23n^2 + 12n$

18. $3^n + 1$ is divisible by 4.

19. $2^n + 2n^2$ is divisible by 4.

20. $n^2 - n + 11$ is prime.

21. $n^2 + n + 41$ is prime.

Prove that each statement is true for all positive integers.

22. $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$

23. $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^n} = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$

24. $12^n + 10$ is divisible by 11.

25. $13^n + 11$ is divisible by 12.

26. ARITHMETIC SERIES Use mathematical induction to prove the formula $a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n - 1)d] = \frac{n}{2}[2a_1 + (n - 1)d]$ for the sum of an arithmetic series.

27. GEOMETRIC SERIES Use mathematical induction to prove the formula $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} = \frac{a_1(1 - r^n)}{1 - r}$ for the sum of a finite geometric series.

EXTRA PRACTICE
See pages 916, 936.
Math Online
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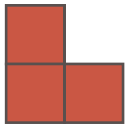
28. PUZZLES Show that a 2^n by 2^n checkerboard with the top right square missing can always be covered by nonoverlapping L-shaped tiles like the one at the right.



H.O.T. Problems

29. OPEN ENDED Write an expression of the form $b^n - 1$ that is divisible by 2 for all positive integers n.

30. CHALLENGE Refer to Example 2. Explain how to use the Binomial Theorem to show that $7^n - 1$ is divisible by 6 for all positive integers n.

31. *Writing in Math* Use the information on page 670 to explain how the concept of a ladder can help you prove statements about numbers.

STANDARDIZED TEST PRACTICE

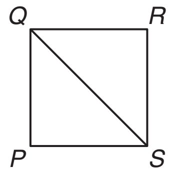
32. ACT/SAT $PQRS$ is a square. What is the ratio of the length of diagonal $\overline{QS}$ to the length of side $\overline{RS}$?

A 2

B $\sqrt{2}$

C 1

D $\frac{\sqrt{2}}{2}$



33. REVIEW The lengths of the bases of an isosceles trapezoid are 15 centimeters and 29 centimeters. If the perimeter of this trapezoid is 94 centimeters, what is the area?

F 500 cm^2

G 515 cm^2

H 528 cm^2

J 550 cm^2

Spiral Review

Expand each power. (Lesson 11-7)

34. $(x + y)^6$

35. $(a - b)^7$

36. $(2x + y)^8$

Find the first three iterates of each function for the given initial value. (Lesson 11-6)

37. $f(x) = 3x - 2, x_0 = 2$

38. $f(x) = 4x^2 - 2, x_0 = 1$

39. BIOLOGY Suppose an amoeba divides into two amoebas once every hour. How long would it take for a single amoeba to become a colony of 4096 amoebas? (Lesson 9-2)